

RESEARCH ARTICLE

Lipschitz inverse shadowing for nonsingular flows

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(Received 00 Month 200x; final version received 00 Month 200x)*

We prove that Lipschitz inverse shadowing for nonsingular flows is equivalent to structural stability.

Keywords: structural stability, shadowing, nonsingular flows

MCS/CCS/AMS Classification/CR Category numbers: MSC 2010: 37C50, 34D30

1. Introduction

The notion of inverse shadowing was introduced by Pilyugin and Corless in [1] and by Kloeden and Ombach in [2]. They defined this notion for diffeomorphisms. Inverse shadowing for flows was first introduced in [3].

It is known that both Lipschitz shadowing and Lipschitz inverse shadowing properties for diffeomorphisms are equivalent to structural stability [4–6]. In [7] the authors proved that structural stability for flows implies inverse shadowing. In fact, they proved that structural stability implies Lipschitz inverse shadowing although they did not use this term in their paper.

We prove that Lipschitz inverse shadowing for flows without rest points implies structural stability.

2. Definitions

Let X be a C^1 vector field on a Riemannian manifold M with metric dist and let Φ be a flow generated by X . We will only consider cases when M is closed and when $M = \mathbb{R}^n$.

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Let d be a (small) positive number.

DEFINITION 2.1 *We say that a mapping $\Psi : \mathbb{R} \times M \rightarrow M$ is a d -method for flow Φ if for any $t \in \mathbb{R}$,*

$$\text{dist}(\Psi(t+s, x), \Phi(s, \Psi(t, x))) < d, \quad s \in [-1, 1], \quad (2.1)$$

and $\Psi(0, x) = x$ for any $x \in M$.

We introduce several classes of d -methods, following [7]. Let Ψ be a d -method. Denote by $(M)^\mathbb{R}$ the set of all functions from \mathbb{R} to M . Consider a mapping

$$\tilde{\Psi} : M \rightarrow (M)^\mathbb{R},$$

defined as

$$\left(\tilde{\Psi}(x)\right)(t) = \Psi(t, x), \quad x \in M, \quad t \in \mathbb{R}.$$

- We say that the d -method Ψ belongs to the class \mathcal{T}_p , if the mapping $\tilde{\Psi}$ is continuous in the pointwise-convergence topology on $(M)^\mathbb{R}$.
- We say that the d -method Ψ belongs to the class \mathcal{T}_o , if the mapping $\tilde{\Psi}$ is continuous in compact-open topology on $(M)^\mathbb{R}$.
- We say that the d -method Ψ belongs to the class \mathcal{T}_c , if it is continuous as a mapping of the form $\mathbb{R} \times M \rightarrow M$.
- We say that the d -method Ψ belongs to the class \mathcal{T}_h , if it is a flow of some C^1 vector field Y that satisfies

$$d_0(X, Y) \leq d.$$

Here d_0 is the C^0 metric on the space of C^1 vector fields on M .

- We say that the d -method Ψ belongs to the class \mathcal{T}_s , if it is smooth as a mapping of the form $\mathbb{R} \times M \rightarrow M$.

Remark 1 Our definition of a d -method is a definition of a family of pseudotrajectories (see the definition of a d -pseudotrajectory of a flow in [4]). If a method belongs to one of the classes defined above, then the dependence of a pseudotrajectory on a point has some continuity (smoothness) properties.

Define Rep as a set of all increasing homeomorphisms $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$. We introduce also the following notation:

$$\text{Rep}(\delta) = \left\{ \alpha \in \text{Rep} \mid \left| \frac{\alpha(t) - \alpha(s)}{t - s} - 1 \right| \leq \delta, \quad t \neq s \right\}.$$

DEFINITION 2.2 *Let \mathcal{T} be one of the classes of methods defined above. We say that a flow Φ has Lipschitz inverse shadowing property (we write $\Phi \in \text{LISP}$ in this case) with respect to the class \mathcal{T} if for any point $p \in M$ there exist constants L, d_0 , such that for any d -method Ψ of the class \mathcal{T} with $d \leq d_0$ there exist a point $\hat{p} \in M$ and a reparametrisation $\alpha \in \text{Rep}(Ld)$ such that*

$$\text{dist}(\Phi(t, p), \Psi(\alpha(t), \hat{p})) < Ld, \quad t \in \mathbb{R}. \quad (2.2)$$

Remark 2 It is easy to see that if δ is small enough and $\alpha \in \text{Rep}(\delta)$ then $\alpha^{-1} \in \text{Rep}(2\delta)$.

Remark 3 Obviously the inclusion $\Phi \in \text{LISP}$ with respect to the class \mathcal{T}_c implies that $\Phi \in \text{LISP}$ with respect to the classes \mathcal{T}_h and \mathcal{T}_s .

Remark 4 It is proved in [7] that the inverse shadowing properties with respect to the classes of methods \mathcal{T}_c , \mathcal{T}_o and \mathcal{T}_p are equivalent. We do not discuss the class \mathcal{T}_h . Further away we write just about Lipschitz inverse shadowing without mentioning a class of methods, always meaning the class \mathcal{T}_s .

3. Main Results

The main result of the work is the following theorem:

THEOREM 3.1 *Let M be a closed manifold and let the flow Φ have no rest points. Then Φ is structurally stable iff $\Phi \in \text{LISP}$.*

4. The idea of the proof

We will use the same idea that the authors of [6] used.

Fix a point $p \in M$. Denote $f = \Phi(1, \cdot)$, $p_k = f^k(p)$, $k \in \mathbb{Z}$, and $A_k = Df(p_k)$, $k \in \mathbb{Z}$. Let $P_k : T_{p_k}M \rightarrow T_{p_k}M$ be the orthogonal projections with kernels $\langle X(p_k) \rangle$ and let V_k be the orthogonal complement to $X(p_k)$. Denote $B_k = P_{k+1}A_k : V_k \rightarrow V_{k+1}$. Consider the following system of difference equations

$$v_{k+1} = B_k v_k + b_{k+1}, \quad k \in \mathbb{Z}. \quad (4.1)$$

Let \mathcal{CR} be the chain-recurrent set of a flow Φ . The following is proved in [6]:

THEOREM 4.1 *Let M be a closed manifold. Suppose that there exists a constant L_1 such that for any point p and any bounded sequence $\{b_k\}_{k \in \mathbb{Z}}$ with entries from the corresponding V_k , equations (4.1) have a solution $\{v_k\}_{k \in \mathbb{Z}}$ with the norm bounded by $L_1 \|b\|$. Then*

- *the set \mathcal{CR} is hyperbolic;*
- *the strong transversality condition is fulfilled.*

It is known (e.g. [8]), that the hyperbolicity of the chain-recurrent set implies Axiom A' . In turn, Axiom A' and strong transversality condition imply structural stability. So the “only if” part of Theorem 3.1 will be proved if we prove that the conditions of the previous theorem are satisfied. The “if” part is proved in [7].

The reason we consider only vector fields without singularities is that unlike in [6] we cannot prove that singularities are isolated in \mathcal{CR} . If we knew that they are then we could easily show that they are hyperbolic and prove Theorem 3.1 holds.

DEFINITION 4.2 *We say that a flow Φ satisfy condition (UB) if*

- (1) *the norms of the derivatives of the flow $\Phi(x, s)$ with respect to initial data for*

$s \in [-1, 1]$ are bounded. I.e., there exists

$$Q_1 = \max_{x \in M, s \in [-1, 1]} \left\| \frac{\partial \Phi(s, x)}{\partial x} \right\|.$$

(2) The lengths of vectors of the vector field X are bounded. I.e., there exists

$$Q_2 = \max_{x \in \mathbb{R}^n} |X(x)|.$$

(3) The reminder in the Taylor expansion of the flow Φ is uniformly bounded.

Let the manifold M be covered by a countable number of open balls V_i of the same radius each admitting a coordinate chart.

There exists a monotonous function $g_1 : [0, \infty) \rightarrow [0, \infty)$ with the following property. If for $x \in M$, $t \in \mathbb{R}$ we fix a coordinate chart of the ball V_i containing the point $\Phi(t, x)$ and denote the representation of the flow Φ in this chart by the same letter Φ , then for any $h_1 \in [-1, 1]$, $h_2 \in T_x M$, $|h_2| < 1$, the following estimate is fulfilled

$$\left| \Phi(t + h_1, x + h_2) - \Phi(t, x) - h_1 X(\Phi(t, x)) - \frac{\partial \Phi(t, x)}{\partial x} h_2 \right| \leq g_1(|h_1| + |h_2|),$$

where $g_1(|h_1| + |h_2|)/(|h_1| + |h_2|)$ tends to 0 uniformly in x for $(h_1, h_2) \rightarrow 0$.

Here we assume that all the points $\Phi(t + h_1, x + h_2)$ belong to V_i , otherwise we can reparametrize the flow globally.

A flow generated by C^1 vector field on a closed manifold satisfies this condition.

At first we prove the solvability of equations that are different from equations (4.1):

Statement 4.3 Let the flow have Lipschitz inverse shadowing property and satisfy condition (UB). Then there exists a constant L_1 such that for any point $p \in M$ and any inhomogeneity $\{z_k\}_{k \in \mathbb{Z}}$ that satisfies $|z_k| \leq 1$, $k \in \mathbb{Z}$, for any natural N there exists a sequence of real numbers $\{s_k\}_{k \in [-N, N]}$ such that the system of equations

$$x_{k+1} = A_k x_k + X(p_{k+1}) s_k + z_{k+1}, \quad k \in [-N, N-1] \quad (4.2)$$

has a solution $\{x_k^{(N)}\}_{k \in [-N, N]}$ such that $|x_k^{(N)}| \leq L_1$, $k \in [-N, N]$.

The proof of this statement is the main difficulty and is given later. Now we show how the solvability of equations (4.2) implies the solvability of equations (4.1). The proof of the next corollary is similar to the proof of Lemma 2 from [6].

COROLLARY 4.4 For any sequence $\{b_k\}_{k \in \mathbb{Z}}$ that satisfies $|b_k| \leq 1$ and $b_k \in V_k$ for all integer k there exists a solution $\{v_k\}_{k \in \mathbb{Z}}$ of system of equations (4.1) such that $|v_k| \leq L_1$.

Proof: Take $z_k = b_k$ in equations (4.2). Statement 4.3 guarantees that there exists a constant L_1 such that for any integer N there exists a sequence $\{s_k\}_{k \in [-N, N]}$ such that system of equations (4.2) has a solution $\{x_k\}_{k \in [-N, N]}$ with the norm bounded by L_1 .

Fix $k \in [-N, N]$ Note that $A_k X(p_k) = X(p_{k+1})$. The definition of the projections P_k implies the inclusion $(\text{Id} - P_k)v \in \langle X(p_k) \rangle$ for any $v \in T_{p_k} M$. Thus $A_k(\text{Id} - P_k) = 0$. Now

we have the equality

$$P_{k+1}A_k = P_{k+1}A_kP_k.$$

Multiply equalities (4.2) by P_{k+1} :

$$\begin{aligned} P_{k+1}x_{k+1} &= P_{k+1}A_kx_k + P_{k+1}X(p_{k+1})s_k + P_{k+1}z_{k+1} = \\ &= P_{k+1}A_kP_kx_k + b_{k+1}, \quad k \in [-N, N-1]. \end{aligned}$$

Thus the sequence $v_k^{(N)} = P_kx_k$ is a solution of equations (4.1) for a finite number of indices. Now we can pass to the limit as $N \rightarrow \infty$. Due to the boundedness, $v_k^{(N)}$ has a limit v_k whose norm is also bounded by L_1 . ■

5. Proof of Statement 4.3 for $M = \mathbb{R}^n$

At first we prove Statement 4.3 when $M = \mathbb{R}^n$ and the vector field X satisfies condition (UB). This will allow us to demonstrate the idea of the proof with less amount of technical details.

Fix a natural N . We are going to use a method from [5]. We will “inscribe” equations into a method and use the Lipschitz inverse shadowing property.

Let us construct a method of the class \mathcal{T}_s that contains the equations “inside”.

Let L, d_0 be constants from the definition of the LISP, and d be a positive number which we decrease later.

Fix positive numbers r and τ . Assume that τ satisfies $100\tau < 1 - \tau$ and $g_1(s) < s$, $s \in [0, \tau]$. Let $\gamma : [-\tau, \tau] \times [0, r] \rightarrow [0, 1]$ be a function with the following properties (see Fig. 1)

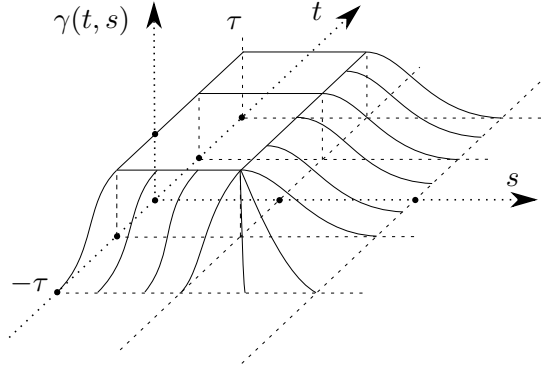
- $\gamma(-\tau, \cdot) = 0$;
- $\gamma(\cdot, r) = 0$;
- $\gamma(t, s) = 1$, $t \in [-\frac{\tau}{2}, \tau]$, $s \in [0, \frac{r}{2}]$;
- $\gamma(t, s) \in (0, 1)$, $(t, s) \notin [-\frac{\tau}{2}, \tau] \times [0, \frac{r}{2}]$;
- γ is smooth as a function of two arguments.

Let $\tilde{g}(d)$ be a function such that it tends to zero faster than d and satisfies $g_1(\tilde{g}(d))/d \rightarrow 0$, $d \rightarrow 0$.

5.1. Construction of the method

Let \varkappa be either 0 or 1 and k be an integer from $[-N, N]$. Suppose that the functions $\Psi_\varkappa(k, \cdot)$, $\Psi_\varkappa(k + \tau, \cdot)$ are already defined. Denote

$$\tilde{p}_k = \Psi_\varkappa(k, p_{-N}), \quad \tilde{A}_k = \frac{\partial \Phi(1 - \tau, \Psi_\varkappa(k + \tau, p_{-N}))}{\partial x},$$

Figure 1. The graph of γ .

$$\hat{p}_{k+1} = \Phi(1 - \tau, \Psi_{\varkappa}(k + \tau)), \quad \hat{p}_{-N} = p_{-N}.$$

If $k < N$, define the function $\Omega_{\varkappa, k+1} : [-\tau, \tau] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ in the following way:

$$\begin{aligned} \Omega_{\varkappa, k+1}(s, v) = & \hat{p}_{k+1} + \varkappa \tilde{A}_k (\Psi_{\varkappa}(k, p_{-N} + v) - \tilde{p}_k + dz_k) + \\ & + \varkappa X(\hat{p}_{k+1})s + dz_{k+1} - \varkappa d\tilde{A}_k z_k. \end{aligned}$$

We also define the interpolation function $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [-\tau, \tau] \rightarrow \mathbb{R}^n$:

$$\Gamma(x, y, v, s) = \gamma(s, |v|)x + (1 - \gamma(s, |v|))y.$$

Define the method Ψ_{\varkappa} in the following way (see Fig. 2):

$$\begin{aligned} \Psi_{\varkappa}(t, x) &= \Phi(t, x), \quad t \in \mathbb{R}, \quad x \notin B_r(p_{-N}); \\ \Psi_{\varkappa}(t, x) &= \Phi(t, x), \quad t \in (-\infty, 1 - \tau], \quad x \in \mathbb{R}^n; \\ \Psi_{\varkappa}(k + s, p_{-N} + v) &= \Gamma(\Omega_{\varkappa, k}(s, v), \Phi(1 - \tau + s, \Psi_{\varkappa}(k - 1 + \tau, p_{-N} + v)), v, s), \\ k &\in [-N + 1, N], \quad s \in [-\tau, \tau], \quad v \in B_r(0); \\ \Psi_{\varkappa}(k + \tau + s, p_{-N} + v) &= \Phi(s, \Psi_{\varkappa}(k + \tau, p_{-N} + v)), \\ k &\in [-N + 1, N - 1], \quad s \in [0, 1 - 2\tau], \quad v \in B_r(0); \\ \Psi_{\varkappa}(N + \tau + s, p_{-N} + v) &= \Phi(s, \Psi_{\varkappa}(N + \tau, p_{-N} + v)), \quad s \in [0, \infty), \quad v \in B_r(0). \end{aligned}$$

Here $B_r(x)$ is the ball of radius r centered at x .

It is easy to see that the mapping we just defined is smooth. Note that the mapping is a perturbation of the flow only at points x that are close to p_{-N} . We emphasize that the mapping Ψ_{\varkappa} depends both on d and r . Note also that \tilde{p}_k and \hat{p}_k do not depend on \varkappa :

$$\tilde{p}_k = \Psi_0(k, p_{-N}) = \Psi_1(k, p_{-N}) = \hat{p}_k + z_k, \quad k \in [-N, N].$$

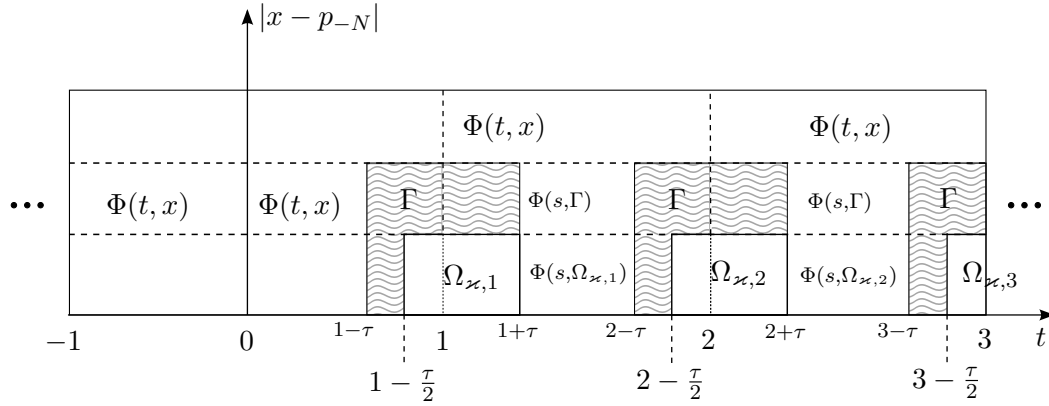


Figure 2. Schematic picture of the method Ψ_ξ . We added N to the numbers below t axis for the ease of display.

Thus we may rewrite the formula for $\Omega_{\xi,k+1}$ in the following way:

$$\begin{aligned} \Omega_{\xi,k+1}(s, v) = & \hat{p}_{k+1} + \xi \tilde{A}_k (\Psi_\xi(k, p_{-N} + v) - \hat{p}_k) + \\ & + \xi X(\hat{p}_{k+1})s + dz_{k+1}, \quad k \in [-N, N-1]. \end{aligned} \quad (5.1)$$

We also define a supplementary mapping $\Theta_\xi(t, x) = \Psi_\xi(t + N, x)$. We need it because we have $t = 0$ in the definition of a d -method and Lipschitz inverse shadowing.

5.2. d -method conditions verification

Let us prove that the mapping Θ_ξ that we have constructed is a $C_\xi(d)$ -method for some positive $C_\xi(d)$ that is independent of N . It is obvious that $\Theta_\xi(0, \cdot) = \text{Id}$. We need to check that inequalities (2.1) are satisfied. The mapping Θ_ξ satisfies them iff Ψ_ξ does. Therefore it is enough to estimate the following value for $t \in \mathbb{R}$, $s \in [-1, 1]$:

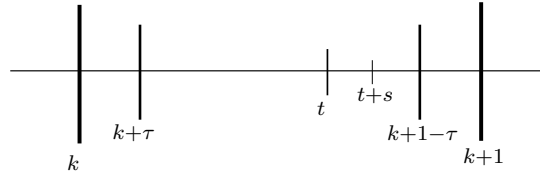
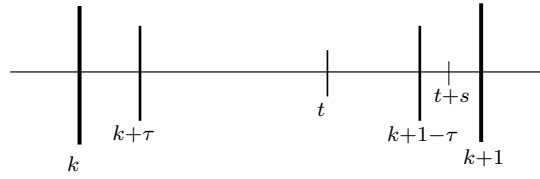
$$\delta_{t,s} = |\Psi_\xi(t + s, p_{-N}) - \Phi(s, \Psi_\xi(t, p_{-N}))|.$$

Since the value of the method $\Psi_\xi(u, p_{-N})$ for $u \in [-\infty, 1 - \tau]$ is equal to $\Phi(u, p_{-N})$, we need to estimate $\delta_{t,s}$ only for $t, t + s \in [1 - \tau, +\infty]$. We will only consider the case of $s \geq 0$, because for $s < 0$ the estimates could be written in a similar way.

We call sections the sets $[k - \tau, k + \tau]$, $k \geq 1$. Note that for t that is outside any section the following is true: if we increase s from 0 to 1 then $t + s$ will cross only one section for a set of s of nonzero measure. If t is in a section then the number of sections we will cross is equal to 2.

Let k be the biggest integer that is less than t . I.e., $k = \lfloor t \rfloor$. Consider the following cases

- t and $t + s$ are outside sections, $s < 2\tau$;
- t are outside sections, $t + s$ is inside a section;

Figure 3. t and $t + s$ are outside sections, $s < 2\tau$ Figure 4. t is outside sections, $t + s$ is inside a section

- t and $t + s$ are outside sections, $s > 2\tau$;
- t is inside a section, $t + s$ is inside a section, $s < 2\tau$;
- t is inside a section, $t + s$ is outside sections;
- t is inside a section, $t + s$ is inside a section, $s > 2\tau$.

5.2.1. t and $t + s$ are outside sections, $s < 2\tau$ (Fig. 3)

In this case

$$\Psi_{\varkappa}(t + s, p_{-N}) = \Phi(s, \Psi_{\varkappa}(t, p_{-N})) = \Phi(t + s - k - \tau, \Psi_{\varkappa}(k + \tau, p_{-N})),$$

thus $\delta_{t,s} = 0$.

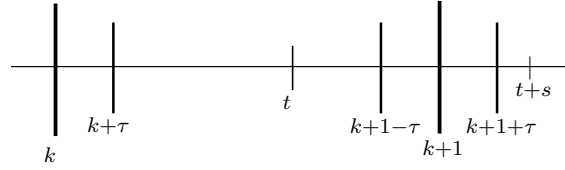
5.2.2. t is outside sections, $t + s$ is inside a section (Fig. 4)

Note that

$$|\Gamma(x, y, v, s) - z| \leq |x - z| + |y - z|, \quad x, y, z \in \mathbb{R}^n. \quad (5.2)$$

Denote $u = t + s - (k + \tau)$ and

$$Q_3 = \sup_{x \in \mathbb{R}^n, u \in [-1, 1] \setminus \{0\}} \frac{|\Phi(u, x) - x|}{u}.$$

Figure 5. t and $t + s$ are outside sections, $s > 2\tau$

Then

$$\begin{aligned} \delta_{t,s} &= |\Psi_{\varkappa}(k+1+t+s-(k+1)), p_{-N}) - \Phi(s, \Phi(t-(k+\tau), \Psi_{\varkappa}(k+\tau, p_{-N})))| = \\ &= |\Gamma(\Omega_{\varkappa, k+1}(t+s-(k+1), 0), \Phi(u, \Psi_{\varkappa}(1+\tau, p_{-N}))) - \Phi(u, \Psi_{\varkappa}(k+\tau, p_{-N}))| \leq \\ &\leq |\Phi(1-\tau, \Psi_{\varkappa}(k+\tau, p_{-N})) + \varkappa X(\Phi(1-\tau, \Psi_{\varkappa}(k+\tau, p_{-N}))) + \\ &\quad - \varkappa \tilde{A}_k dz_k + dz_{k+1} - \Phi(u, \Psi_{\varkappa}(k+\tau, p_{-N}))|. \end{aligned}$$

It follows that

- if $\varkappa = 1$ then

$$\delta_{t,s} \leq g_1(u) + Q_1 d + d \leq g_1(\tau) + Q_1 d + d;$$

- if $\varkappa = 0$ then

$$\delta_{t,s} \leq Q_3 u + Q_1 d + d \leq Q_3 \tau + d.$$

5.2.3. t and $t + s$ are outside sections, $s > 2\tau$ (Fig. 5)

Denote

$$Q_4 = \sup_{x, y \in \mathbb{R}^n, x \neq y, s \in [-1, 1]} \frac{|\Phi(s, x) - \Phi(s, y)|}{|x - y|}$$

and $u = k + 1 + \tau$. Then

$$\begin{aligned} \delta_{t,s} &= |\Phi(t+s-u, \Psi_{\varkappa}(u, p_{-N})) - \Phi(t+s-u, \Phi(u-t, \Psi_{\varkappa}(t, p_{-N})))| \leq \\ &\leq Q_4 |\Psi_{\varkappa}(u, p_{-N}) - \Phi(u-t, \Psi_{\varkappa}(t, p_{-N}))|. \end{aligned}$$

Now we may use the equality from the previous case. We get the following:

- if $\varkappa = 1$ then

$$\delta_{t,s} \leq Q_4(g_1(\tau) + Q_1 d + d);$$

- if $\varkappa = 0$ then

$$\delta_{t,s} \leq Q_4(Q_3 \tau + d).$$

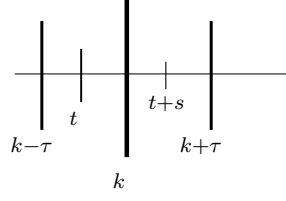


Figure 6. t is inside a section, $t + s$ is inside a section, $s < 2\tau$

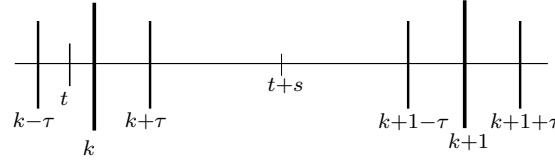


Figure 7. t is inside a section, $t + s$ outside sections

5.2.4. t is inside a section, $t + s$ is inside a section, $s < 2\tau$ (Fig. 6)

Now let k be an integer that is the closest to t . I.e., $k = [t]$.

Denote $u = k - \tau$. Using the estimate from case 5.2.2 we may write

$$\begin{aligned} \delta_{t,s} &\leq |\Psi_{\varkappa}(t + s, p_{-N}) - \Phi(t + s - u, \Psi_{\varkappa}(u, p_{-N}))| + \\ &\quad + |\Phi(t + s - u, \Psi_{\varkappa}(u, p_{-N})) - \Phi(s, \Psi_{\varkappa}(t, p))|. \end{aligned}$$

Thus

- if $\varkappa = 1$ then

$$\begin{aligned} \delta_{t,s} &\leq g_1(t + s - u) + Q_1d + d + sQ_4 |\Phi(t - u, \Psi_{\varkappa}(u, p_{-N})) - \Psi_{\varkappa}(t, p_{-N})| \leq \\ &\leq g_1(\tau) + Q_1d + d + sQ_4(g_1(t - u) + Q_1d + d) \leq \\ &\leq g_1(\tau) + Q_1d + d + Q_4(g_1(\tau) + Q_1d + d); \end{aligned}$$

- if $\varkappa = 0$ then

$$\begin{aligned} \delta_{t,s} &\leq Q_3\tau + d + sQ_4(Q_3(t - u) + d) \leq \\ &\leq Q_3\tau + d + Q_4(Q_3\tau + d). \end{aligned}$$

5.2.5. t is inside a section, $t + s$ outside sections (Fig. 7)

The next estimate follows from the previous one like in case 5.2.3:



Figure 8. t is inside a section, $t + s$ is inside a section, $s > 2\tau$

- if $\varkappa = 1$ then

$$\delta_{t,s} \leq Q_4(g_1(\tau) + Q_1d + d + Q_4(g_1(\tau) + Q_1d + d));$$

- if $\varkappa = 0$ then

$$\delta_{t,s} \leq Q_4(Q_3\tau + d + Q_4(Q_3\tau + d)).$$

5.2.6. t is inside a section, $t + s$ is inside a section, $s > 2\tau$ (Fig. 8)

Take

$$\begin{aligned} A &= |\Omega_{\varkappa, k+1}(k+1 - (t+s), 0) - \Phi(t+s - (t+\tau), \Psi_{\varkappa}(k+\tau, p))|, \\ B &= |\Phi(t+s - (t+\tau), \Psi_{\varkappa}(k+\tau, p)) - \Phi(s, \Psi_{\varkappa}(t, p_{-N}))|. \end{aligned}$$

Then

$$\delta_{t,s} \leq A + B.$$

We estimate the summands separately. By definition,

$$\begin{aligned} \Psi_{\varkappa}(t+s, p) &= \Gamma\left(\Omega_{\varkappa, k+1}(k+1 - (t+s), 0), \right. \\ &\quad \left. \Phi(t+s - (k+\tau), \Psi_{\varkappa}(k+\tau, p_{-N}))\right). \end{aligned}$$

Note that

$$\Phi(s, \Psi_{\varkappa}(t, p_{-N})) = \Phi(s - (k+\tau - t), \Phi(t+\tau - t, \Psi_{\varkappa}(t, p))).$$

Now we use condition (UB):

$$\begin{aligned} \Phi(k+\tau - t, \Psi_{\varkappa}(t, p)) &= \Phi(k+\tau - t, \hat{p}_k + \varkappa X(\hat{p}_k)(t-k) + dz_k - \varkappa dA_k z_{k-1}) = \\ &= \hat{p}_k + X(\hat{p}_k)(k+\tau - t) + \varkappa X(\hat{p}_k)(t-k) + dz_k - \varkappa dA_k z_{k-1} + g_1(d+\tau). \end{aligned}$$

Moreover

$$\Psi_{\varkappa}(k+\tau, p_{-N}) = \tilde{p}_k + \varkappa X(\hat{p}_k)\tau + dz_k - \varkappa dA_k z_{k-1}.$$

Thus we get

$$\Psi_{\varkappa}(k + \tau, p_{-N}) - \Phi(t + \tau - t, \Psi_{\varkappa}(t, p)) = (1 - \varkappa)X(\hat{p}_k)(t - k) + g_1(\tau + d).$$

Consequently

$$\begin{aligned} B &\leq \left\| \frac{\partial \Phi(t + s - (t + \tau), \cdot)}{\partial x} (\Psi_{\varkappa}(k + \tau, p_{-N})) \right\| \cdot \\ &\quad \cdot |\Psi_{\varkappa}(k + \tau, p_{-N}) - \Phi(t + \tau - t, \Psi_{\varkappa}(t, p))| + \\ &\quad + g_1(|\Psi_{\varkappa}(k + \tau, p_{-N}) - \Phi(t + \tau - t, \Psi_{\varkappa}(t, p))|) \leq \\ &\leq Q_1((1 - \varkappa)Q_2\tau + g_1(\tau + d)) + g_1(\tau + d). \end{aligned}$$

We may estimate A like we did in case 5.2.2. Therefore

- if $\varkappa = 1$ then

$$\delta t, s \leq g_1(\tau) + Q_1d + d + Q_1(g_1(\tau + d)) + g_1(\tau + d);$$

- if $\varkappa = 0$ then

$$\delta t, s \leq Q_3\tau + dQ_1(Q_2\tau + g_1(\tau + d)) + g_1(\tau + d).$$

5.2.7. The final estimate

Take $\tau = \tilde{g}(d)$. Our estimates imply that Ψ_{\varkappa} is a $C_{\varkappa}(d)$ -method with

- if $\varkappa = 1$ then

$$C_1(d) = K_1d + K_2g_1(\tilde{g}(d));$$

- if $\varkappa = 0$ then

$$C_0(d) = K_3d + K_4\tilde{g}(d).$$

Here K_i , $i = 1..4$ are some constants that are independent of N . This means that we can decrease d so that

$$LC_{\varkappa}(d) < \min \left(d_0, \frac{r}{2} \right). \quad (5.3)$$

Moreover if $\varkappa = 1$ then we can guarantee that the following inequality is satisfied

$$LC_1(d) \leq \frac{\tau}{4N}.$$

5.3. Shadowing of an exact trajectory by a trajectory of the method

At first we estimate the value $\delta_k = |\tilde{p}_k - p_k|$, $k \in [-N, N]$. For $k = 0$ we obviously have $|p_{-N} - \tilde{p}_{-N}| \leq LC_0(d)$. The Lipschitz inverse shadowing property guarantees that there exists a point p^* and a reparametrization $\alpha \in \text{Rep}(LC_0(d))$ such that

$$|\Phi(t, p_{-N}) - \Theta_0(\alpha(t), p^*)| \leq LC_0(d) \leq \frac{r}{2}, \quad t \in \mathbb{R}.$$

Fix $k \in [-N, N]$. Denote

$$a_k = \alpha(k + N) - N, \quad x_k = \Theta_0(a_k + N, p^*) = \Psi_0(a_k, p^*).$$

Thus

$$\delta_k \leq |p_k - x_k| + |x_k - \tilde{p}_k| \leq LC_0(d) + |x_k - \tilde{p}_k|.$$

Let $k > -N$. Consider the following cases:

- if $a_k > k + \tau$ then

$$\Psi_0(a_k, p^*) = \Phi(a_k - (k + \tau), \Psi_0(k, p^*)) = \Phi(a_k - (k + \tau), \tilde{p}_k).$$

Hence

$$|x_k - \tilde{p}_k| = |\Phi(a_k - (k + \tau), \tilde{p}_k) - \tilde{p}_k| \leq Q_3(a - k - \tau) \leq 2Q_3LC_0(d).$$

- If $a_k < k - \tau$ then

$$\begin{aligned} |x_k - \tilde{p}_k| &= |\Phi(a_k - (k - 1 + \tau), \Psi_0(k - 1 + \tau, p^*)) - \\ &\quad - \Phi(1 - \tau, \Psi_0(k - 1 + \tau, p_{-N})) - dz_k| = \\ &= |\Phi(a - (k - 1 + \tau), \tilde{p}_{k-1}) - \Phi(1 - \tau, \tilde{p}_{k-1})| \leq Q_3(a_k - k) \leq Q_3LC_0(d). \end{aligned}$$

- If $a_k \in [k - \tau, k + \tau]$ then denote $u = k - 1 + \tau$. Thus

$$\begin{aligned} |x_k - \tilde{p}_k| &= |\Gamma(\Phi(a_k - u, \Psi_0(u, p^*)), \tilde{p}_k) - \tilde{p}_k| \leq \\ &\leq |\Phi(a_k - u, \Psi_0(u, p^*)) - \tilde{p}_k|. \end{aligned}$$

Now we can estimate the above value similarly to the previous case.

Therefore we just proved that

$$|\tilde{p}_k - p_k| \leq K_5 d, \quad k \in [-N, N], \quad (5.4)$$

where K_5 is a constant that is independent of N .

Now we prove the solvability of equations using Ψ_1 . To emphasize the dependence on d we will further write upper index (d) . Lipschitz inverse shadowing guarantees that there

exists a point $\hat{p}^{(d)}$ and a reparametrization $\beta^{(d)} \in \text{Rep}(LC_1(d))$ such that

$$\left| \Phi(t, p_{-N}) - \Theta_1^{(d)}(\beta^{(d)}(t), \hat{p}^{(d)}) \right| \leq LC_1(d) \leq \frac{r}{2}, \quad t \in \mathbb{R}.$$

Choose d small enough for inequality (5.3) to work. Then

$$\left| \beta^{(d)}(k) - k \right| \leq LC_1(d)k \leq \frac{\tau}{2}, \quad k \in [0, 2N]. \quad (5.5)$$

Denote $v^{(d)} = -p_{-N} + \hat{p}^{(d)}$,

$$\sigma_k^{(d)} = \beta^{(d)}(k + N) - k, \quad y_k^{(d)} = \Theta_1^{(d)}(\beta^{(d)}(k + N), p_{-N} + v^{(d)}), \quad k \in [-N, N].$$

Inequality (5.5) and the definition of the function γ implies the following equalities:

$$y_k^{(d)} = \Omega_{1,k}^{(d)}(\sigma_k^{(d)}, v^{(d)}), \quad k \in [-N, N].$$

Set $W_k^{(d)} = \Psi_1^{(d)}(k, p_{-N} + v^{(d)}) - \hat{p}_k^{(d)}$ for $k \in [-N, N]$. Using formula (5.1) we conclude that the sequence $W_k^{(d)}$ solves the following equations:

$$W_{k+1}^{(d)} = \tilde{A}_k^{(d)} W_k^{(d)} + X(\hat{p}_{k+1}^{(d)}) \sigma_{k+1}^{(d)} + dz_{k+1}, \quad k \in [-N, N-1].$$

Divide them by d and take

$$w_k^{(d)} = (y_k^{(d)} - \hat{p}_k^{(d)})/d, \quad s_k^{(d)} = \sigma_k^{(d)}/d, \quad k \in [-N, N].$$

Thus

$$w_{k+1}^{(d)} = \tilde{A}_k^{(d)} w_k^{(d)} + X(\hat{p}_{k+1}^{(d)}) s_{k+1}^{(d)} + z_{k+1}, \quad k \in [-N, N-1]. \quad (5.6)$$

Now we estimate $|W_k^{(d)}|$:

$$\left| W_k^{(d)} \right| \leq \left| \Psi_1^{(d)}(k, p_{-N} + v^{(d)}) - y_k^{(d)} \right| + \left| y_k^{(d)} - \hat{p}_k^{(d)} \right|, \quad k \in [-N, N]. \quad (5.7)$$

At first we deal with the first summand:

$$\begin{aligned} \left| \Psi_1^{(d)}(k, p_{-N} + v^{(d)}) - y_k^{(d)} \right| &= \left| \Omega_{1,k}^{(d)}(0, v^{(d)}) - \Omega_{1,k}^{(d)}(\sigma_k^{(d)}, v^{(d)}) \right| = \\ &= \left| X(\hat{p}_{k-1}^{(d)}) \sigma_k^{(d)} \right| \leq Q_2 LC_1(d), \quad k \in [-N, N]. \end{aligned}$$

We can write estimates for the second summand of the right-hand side of (5.7), using inequality (5.4):

$$\begin{aligned}
& \left| y_k^{(d)} - \hat{p}_k^{(d)} \right| \leq \left| p_k - \hat{p}_k^{(d)} \right| + \left| y_k^{(d)} - p_k \right| \leq \\
& \leq LC_1(d) + |\Phi(1, p_{k-1}) - \Phi(1 - \tau, \Psi_1(k + \tau, p_{-N}))| \leq \\
& \leq \left| \Phi(1, p_{k-1}) - \Phi(1 - \tau, \tilde{p}_k^{(d)}) \right| + \left| \Phi(1 - \tau, \tilde{p}_k^{(d)}) - \Phi(1 - \tau, \Psi_1(k - 1 + \tau, p_{-N})) \right| \leq \\
& \leq \left| X(\hat{p}_{k-1}^{(d)})\tau + A_k(p_{k-1} - \tilde{p}_{k-1}^{(d)}) + g_1(|\tau| + |p_{k-1} - \tilde{p}_{k-1}^{(d)}|) \right| + \\
& + Q_4 \left| \tilde{p}_{k-1}^{(d)} - \Psi_1(k - 1 + \tau, p_{-N}) \right| \leq Q_2\tau + 2Q_1K_5d + Q_4Q_2\tau \leq \\
& \leq Q_2LC_1(d) + 2Q_1K_5d + Q_4Q_2LC_1(d), \quad k \in [-N + 1, N].
\end{aligned}$$

Denote

$$L_1 = \frac{1}{d} (Q_2LC_1(d) + Q_2LC_1(d) + 2Q_1K_5d + Q_4Q_2LC_1(d)).$$

Therefore

$$\left| w_k^{(d)} \right| = \left| \frac{W_k^{(d)}}{d} \right| \leq L_1, \quad k \in [-N, N].$$

Then up to taking a subsequence, there exists the limit

$$w_k^* = \lim_{d \rightarrow 0} w_k^{(d)}, \quad |w_k^*| \leq L_1, \quad k \in [-N, N].$$

Since by assumption the vector field does not have rest points, the values $|X(x)|$ are bounded both from above and from 0 for any $x \in \mathbb{R}^n$. The values $\left| X(\hat{p}_{k+1}^{(d)})s_{k+1}^{(d)} \right|$ are also bounded due to the fact that all other terms in equality (5.6) are bounded. Hence the values $s_{k+1}^{(d)}$ are also bounded and converge to s_{k+1}^* . Summarizing all these we just have proved that the sequence $\{w_k^*\}$ satisfies the desired equations

$$\begin{aligned}
w_{k+1}^* &= \lim_{d \rightarrow 0} w_{k+1}^{(d)} = \lim_{d \rightarrow 0} \tilde{A}_k^{(d)} w_k^{(d)} + \lim_{d \rightarrow 0} X(\hat{p}_{k+1}^{(d)}) s_{k+1}^{(d)} + z_{k+1} = \\
&= A_k w_k^* + X(p_{k+1}) s_{k+1}^* + z_{k+1}, \quad k \in [-N, N - 1]
\end{aligned}$$

and is bounded by a constant L_1 that is independent of N .

6. Proof of Statement 4.3 for the case of a closed manifold

Fix a natural N . We again will construct a method of the class \mathcal{T}_s that will “contain” the equations. We use the same notation as in the beginning of section 5.

Let r_1 be a positive number such that for any x the mappings \exp_x , \exp_x^{-1} are defined and are diffeomorphisms on $B_{r_1}(x) \subset M$ and $B_{r_1}(0) \subset T_x M$ correspondingly. Without loss of generality we may assume that the radius r_1 is small enough so that the exponential mappings and their inverses distort distances less than twice:

$$\begin{aligned} |\exp_x^{-1}(y_1) - \exp_x^{-1}(y_2)| &\leq 2 \operatorname{dist}(y_1, y_2), \quad y_1, y_2 \in B_{r_1}(x) \subset M; \\ \operatorname{dist}(\exp_x(\tilde{y}_1), \exp_x(\tilde{y}_2)) &\leq 2 |\tilde{y}_1 - \tilde{y}_2|, \quad \tilde{y}_1, \tilde{y}_2 \in B_{r_1}(0) \subset T_x M. \end{aligned}$$

We define the method like we did in section 5.1 with the exception that now we should take into account that we work on a manifold.

Let \varkappa be equal to either 0 or 1 and k be an integer from the interval $[-N, N]$. Assume that the functions $\Psi_\varkappa(k, \cdot)$, $\Psi_\varkappa(k + \tau, \cdot)$ are already defined. Denote

$$\begin{aligned} \tilde{p}_k &= \Psi_\varkappa(k, p_{-N}), \quad \tilde{A}_k = \frac{\partial \Phi(1 - \tau, \Psi_\varkappa(k + \tau, p_{-N}))}{\partial x}, \\ \hat{p}_{k+1} &= \Phi(1 - \tau, \Psi_\varkappa(k + \tau)), \quad \hat{p}_{-N} = p_{-N}. \end{aligned}$$

If $k < N$ we define the function $\Omega_{\varkappa, k+1} : [-\tau, \tau] \times T_{p_{-N}} M \rightarrow T_{p_{k+1}} M$ in the following way:

$$\begin{aligned} \Omega_{\varkappa, k+1}(s, v) &= \varkappa \tilde{A}_k \left(\exp_{\tilde{p}_k}^{-1} (\Psi_\varkappa(k, \exp_{p_{-N}}(v))) + dz_k \right) + \\ &\quad + \varkappa X(\hat{p}_{k+1})s + dz_{k+1} - \varkappa d\tilde{A}_k z_k. \end{aligned}$$

We define also the interpolation function $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [-\tau, \tau] \rightarrow \mathbb{R}^n$:

$$\Gamma(x, y, v, s) = \gamma(s, |v|)x + (1 - \gamma(s, |v|))y.$$

Fix a positive r that is less than r_1 . Define the method Ψ_\varkappa in the following way:

$$\begin{aligned} \Psi_\varkappa(t, x) &= \Phi(t, x), \quad t \in \mathbb{R}, \quad x \notin B_r(p_{-N}) \subset M; \\ \Psi_\varkappa(t, x) &= \Phi(t, x), \quad t \in (-\infty, 1 - \tau], \quad x \in \mathbb{R}^n. \\ \Psi_\varkappa(k + s, \exp_{p_{-N}}(v)) &= \\ &= \exp_{\hat{p}_k} \left(\Gamma \left(\Omega_{\varkappa, k}(s, v), \exp_{\hat{p}_k}^{-1} (\Phi(1 - \tau + s, \Psi_\varkappa(k - 1 + \tau, \exp_{p_{-N}}(v)))) \right), v, s \right), \\ &\quad k \in [-N + 1, N], \quad s \in [-\tau, \tau], \quad v \in B_r(0) \subset T_{p_{-N}} M; \\ \Psi_\varkappa(k + \tau + s, \exp_{p_{-N}}(v)) &= \Phi(s, \Psi_\varkappa(k + \tau, \exp_{p_{-N}}(v))), \\ &\quad k \in [-N + 1, N - 1], \quad s \in [0, 1 - 2\tau], \quad v \in B_r(0) \subset T_{p_{-N}} M; \\ \Psi_\varkappa(N + \tau + s, \exp_{p_{-N}}(v)) &= \Phi(s, \Psi_\varkappa(k + \tau, \exp_{p_{-N}}(v))), \\ &\quad s \in [0, \infty), \quad v \in B_r(0) \subset T_{p_{-N}} M. \end{aligned}$$

Unlike the case of $M = \mathbb{R}^n$ here we need some additional assumptions for the definition to be correct. We need all the mappings \exp and \exp^{-1} that were used to be defined. It

is easy to see that it is enough to guarantee that the following inequalities are satisfied:

$$\begin{aligned} \text{dist}(\tilde{p}_k, \Psi_{\varkappa}(k, \exp_{p_{-N}}(v))) &\leq r_1, \quad k \in [-N, N]; \\ \text{dist}(\hat{p}_k, \Phi(1 - \tau + s, \Psi_{\varkappa}(k - 1 + \tau, \exp_{p_{-N}}(v)))) &\leq r_1, \quad k \in [-N, N - 1]; \\ \left| \Gamma\left(\Omega_{\varkappa, k}(s, v), \exp_{\hat{p}_k}^{-1}(\Phi(1 - \tau + s, \Psi_{\varkappa}(k - 1 + \tau, \exp_{p_{-N}}(v))))\right), v, s \right| &\leq r_1, \\ k &\in [-N, N - 1]. \end{aligned}$$

One can verify that the right-hand sides of these inequalities can be bounded from above by $10Q_1^{2N}(d + r + \tau Q_2)$. This means that it is enough to take d, r and τ small enough.

The rest of the proof is fully analogous to the case of $M = \mathbb{R}^d$.

Acknowledgements

Research was supported by RFBR (project 12-01-00275) and the Chebyshev laboratory (grant of the Russian government N 11.G34.31.0026).

I am grateful to S. Yu. Pilyugin and S. Tikhomirov for their helpful comments and advice and to A. Petrov for discussions.

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